

# A LOWER BOUND LIMITING SOLUTIONS IN THE HYPERBOLIC CASE OF THE GENERALIZED FERMAT EQUATION

BRUCE ZIMOV

ABSTRACT. We find a lower bound for  $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$  limiting any solution in the hyperbolic case of the Generalized Fermat Equation  $x^p + y^q = z^r$ .

## 1. INTRODUCTION

Let  $p, q, r, x, y, z \in \mathbb{Z}$ ,  $(x, y, z) = 1$ , and  $x, y, z \geq 2$ . The hyperbolic case of the Generalized Fermat Equation is

$$x^p + y^q = z^r$$

with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

The absence of solutions for  $p, q, r \geq 3$  has been recently surveyed in [BCDY15]. A lower bound for  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$  means that there are no solutions to the Generalized Fermat Equation with exponents  $(p, q, r)$  below this lower bound. Assuming Baker's Explicit *abc*-conjecture, theorems have been proven with explicit lower bounds for  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ .

In particular, Laishram and Shorey [LS12] showed that

$$\frac{4}{7} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

Chim Shorey and Sinha [CSS19] improved this to

$$\frac{1}{1.72} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

Denote the greatest square-free factor of  $xyz$  as  $G = G(x, y, z) = \prod_{p|xyz} p$ , the radical of  $xyz$ , where  $p$  is a prime. Without relying on any conjecture, Stewart and Yu [SY91] proved the following: There exists an effectively computable positive constant  $c$  such that for all positive integers  $x, y$ , and  $z$  with  $(x, y, z) = 1$ ,  $z > 2$ , and  $x + y = z$ ,

---

*Date:* December 5, 2020.

*2020 Mathematics Subject Classification.* Primary 11D41.

$$\log(z) < G^{2/3 + \frac{c}{\log \log(G)}}$$

Stewart and Yu [SY98][SY01] subsequently improved the upper bound to

$$(1) \quad \log(z) < G^{1/3 + \frac{c}{\log \log(G)}}$$

Wong Chi Ho [Won99] obtained  $c = 15$  for the effectively computable constant in the upper bound. Using this result, we will obtain a lower bound for  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$  in the hyperbolic case of the Generalized Fermat Equation without relying on any conjecture. In this paper, we will prove the following:

**Theorem 1.** *Let  $p, q, r, x, y, z \in \mathbb{Z}$ .*

*If  $(x, y, z) = 1$ ,  $x, y, z \geq 2$ , and  $p, q, r \geq 3$ , then the equation*

$$(2) \quad x^p + y^q = z^r$$

*has no solutions with*

$$(3) \quad \frac{3 \cdot \log \log(z^r)}{\log(z^r)} \frac{1}{\left(1 + \frac{45}{\log \log(G)}\right)} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

*where  $G = G(x, y, z)$ .*

## 2. PROOF OF THEOREM 1

*Proof.* From (2) above,

$$x < z^{r/p}, y < z^{r/q}.$$

By the Main Theorem of Wong's thesis [Won99], we have

$$\log(z^r) < G^{1/3 + \frac{15}{\log \log(G)}}$$

We note that  $G = G(x, y, z) \leq xyz < z^{r\chi}$  where  $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ . Then,

$$\log(z^r) < z^{r\chi \left(1/3 + \frac{15}{\log \log(G)}\right)}$$

$$\frac{\log \log(z^r)}{\log(z^r)} < \chi \left(1/3 + \frac{15}{\log \log(G)}\right)$$

$$\frac{3 \cdot \log \log(z^r)}{\log(z^r)} \frac{1}{\left(1 + \frac{45}{\log \log(G)}\right)} < \chi$$

There is no solution to (2) for  $\chi$  below this lower bound. Hence, the assertion.  $\square$

### 3. COROLLARIES

#### 3.1. Negative lower bound.

**Proposition 1.** *If the lower bound to  $\chi$  is negative, then no  $(p, q, r)$  are excluded for solving (2).*

*Proof.*  $z, r, \chi,$  and  $G$  are positive. There are no  $\chi$  values below any negative lower bound. Hence, no  $(p, q, r)$  are excluded for solving (2) in this case.  $\square$

**Example 1.** *Negative lower bound for  $\chi$ .*

Stewart and Yu [SY01] further improved the upper bound of (1) to

$$z < e^{c \cdot G^{1/3} (\log(G))^3}$$

Applied to (2), we get

$$(4) \quad \log(z^r) < c \cdot G^{1/3} (\log(G))^3$$

We proceed with  $G < z^{r\chi}$  as in our proof of Theorem 1 above.

$$\begin{aligned} \log(z^r) &< c \cdot z^{r\chi/3} (\log(z^{r\chi}))^3 \\ \log\left(\frac{\log(z^r)}{c}\right) &< \frac{\chi}{3} \cdot \log(z^r) + 3 \cdot \log \log(z^{r\chi}) \\ \log \log(z^r) - \log(c) &< \frac{\chi}{3} \cdot \log(z^r) + 3 \cdot \log(\chi) + 3 \cdot \log \log(z^r) \\ -(2 \cdot \log \log(z^r) + \log(c)) \cdot \frac{3}{\log(z^r)} &< \chi + 9 \cdot \frac{\log(\chi)}{\log(z^r)} < \chi \cdot \left(1 + \frac{9}{\log(z^r)}\right) \\ \frac{-3 \cdot (2 \cdot \log \log(z^r) + \log(c))}{\log(z^r) + 9} &< \chi \end{aligned}$$

Clearly the left hand side is negative unless

$$0 < r < \frac{1}{\sqrt{c} \log(z)}$$

Chim Kwok Chi [Chi05] obtained  $c = e^{2.6 \times 10^{44}}$  for the effectively computable constant. Thus,

$$0 < r < \frac{1}{\sqrt{c} \log(z)} < 3$$

Hence, by the proposition, no  $(p, q, r)$  are excluded for solving (2) given (4).

### 3.2. Boundedness of $G$ .

**Corollary 1.1.** Let  $\phi(x) = \frac{3 \cdot \log \log(x)}{\log(x)}$  and  $\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ .

$$(5) \quad \log \log(G) < \frac{45}{\frac{\phi(z^r)}{\chi} - 1}$$

*Proof.* This is a consequence of (3). □

Consider the values of  $\phi(z^r)$  as given in Table 1. Observe that for  $z^r > 2^4$ ,  $\phi(z^r)$  is a decreasing function. Consequently,  $\phi(z^r)$  is at a maximum at a value of 1.1034. If  $p$  and  $q$  go to  $\infty$ , we have

$$\lim_{p, q \rightarrow \infty} \frac{\phi(z^r)}{\chi} = \frac{3 \cdot \log \log(z^r)}{\log(z)}$$

For  $p = q = 3$  at their minimum, we have

$$\frac{\phi(z^r)}{\chi} = \frac{3}{2 \cdot r + 3} \cdot \frac{3 \cdot \log \log(z^r)}{\log(z)}$$

In general, we have

$$\frac{\phi(z^r)}{\chi} = \frac{p \cdot q}{(p + q) \cdot r + p \cdot q} \cdot \frac{3 \cdot \log \log(z^r)}{\log(z)}$$

If  $\frac{\phi(z^r)}{\chi} > 1$  we have an upper bound to  $G$ .

If  $\frac{\phi(z^r)}{\chi} = 1$  then  $G$  is unbounded.

$\frac{\phi(z^r)}{\chi} < 1$  is a contradiction because  $G \geq 30 = 2 \cdot 3 \cdot 5$ .

$z^r$	$\phi(z^r)$
$2^3$	1.0562
$2^4$	1.1034
$3^3$	1.0856
$2^5$	1.0759
$2^6$	1.0281
$3^4$	1.0106
$5^3$	0.9783
$2^7$	0.9765

TABLE 1. First 8 values of  $\phi(z^r)$ . Calculations were done with the Java Math Library.

## ACKNOWLEDGEMENT

The author would like to thank Cameron Stewart for his valuable comments on an earlier version of this paper.

## REFERENCES

- [BCDY15] M.A. Bennett, I. Chen, S.R. Dahmen, and S. Yazdani, *Generalized Fermat equations: A miscellany*, International Journal of Number Theory **11** (2015), no. 1, 1–28, DOI 10.1142/S179304211530001X. ↑1
- [CSS19] Kwok Chi Chim, T.N. Shorey, and Sneha Bala Sinha, *On Baker’s explicit abc-conjecture*, Publicationes Mathematicae Debrecen **94** (2019), 435–453, DOI 10.5486/pmd.2019.8397. ↑1
- [Chi05] Kwok Chi Chim, *New explicit result related to the abc-conjecture*, Masters thesis, Hong Kong University, 2005, <https://1bezone.ust.hk/bib/b863988>. ↑3
- [Won99] Chi Ho Wong, *An explicit result related to the abc-conjecture*, Masters thesis, Hong Kong University, 1999, <https://1bezone.ust.hk/bib/b645947>. ↑2
- [LS12] Shanta Laishram and T.N. Shorey, *Baker’s explicit abc-conjecture and applications*, Acta Arithmetica **155** (2012), no. 4, 419–429, DOI 10.4064/aa155-4-6. ↑1
- [SY91] C. L. Stewart and Kunrui Yu, *On the abc conjecture*, Mathematische Annalen **291** (1991), no. 2, 225–230. ↑1
- [SY98] ———, *On the abc conjecture II*, University of Arizona, 1998. Diophantine Geometry Related to the ABC Conjecture Workshop. ↑2
- [SY01] ———, *On the abc conjecture II*, Duke Mathematical Journal **108** (2001), no. 1, 169–181, DOI 10.1215/S0012-7094-01-10815-6. ↑2, 3

CALIMESA RESEARCH INSTITUTE, 33562 YUCAIPA BLVD 4-321, YUCAIPA, CA 92399, USA

*Email address:* [katcha997@aol.com](mailto:katcha997@aol.com)